Scaling treatment of anisotropic diffusion

1985 J. Phys. A: Math. Gen. 18 L591

(http://iopscience.iop.org/0305-4470/18/10/007)

J. Phys. A: Math. Gen. 18 (1985) L591-L595. Printed in Great Britain

LETTER TO THE EDITOR

Scaling treatment of anisotropic diffusion

R B Stinchcombe

Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

Received 27 March 1985

Abstract. Exact scaling treatments are given of anisotropic diffusion in pure and diluted chains, and in a fractal. A dynamic decimation method is used to obtain detailed dynamic scaling descriptions. The results for chains show crossover to drift or localised behaviour, induced by anisotropy or dilution. The fractal results include bias-induced crossover, with exponent unity, from anomalous diffusion behaviour (exponent $\log_2 5$) to drift behaviour, and scaling corrections (exponent $\log_2 \frac{5}{3}$) arising from rotational anisotropy.

Dynamic behaviour can be strongly modified by critical effects, e.g. those associated with the divergence of a thermal correlation length at an ordinary second-order transition (see e.g. [1]), or a geometrical characteristic length at such transitions as the percolation transition in dilute systems [2-4]. An example is the modification of the power law relating wavevector k and frequency ω for long wavelength spin waves near the percolation transition in a dilute magnet [2-8]. This is closely related to the change over from simple hopping diffusion to an anomalous form at the percolation threshold [9-11]. These related effects have been the subject of much recent discussion using approximate scaling treatments of the dilute system [8], or by treating [8-11] its approximate representation by a non-random fractal [12, 13] or by simulation [14] or other computational methods [15].

The change over between simple-spin wave dispersion, or simple diffusion, and modified ('anomalous') behaviour is an example of a relevant field inducing crossover. The importance and great diversity of crossover effects is well known in static critical behaviour (see, e.g. [16]) but is relatively unexplored in critical dynamics, that in the neighbourhood of the percolation threshold being one of the few examples so far considered in detail. Another currently receiving attention is the class of random walks with various types of relevant perturbation, e.g. interactions [17, 18], 'true' self avoidance [19], or trapping [20]. An important member of this class is the biased random walk, which is equivalent to an anisotropic diffusion problem (with asymmetric transfer rates). For this system, Monte Carlo simulations [21] have shown striking crossover behaviour. Biased diffusion in a one-dimensional model has been recently discussed [22], but little is known about anisotropy effects in critical dynamics in more general situations.

The aim of the present letter is to give an analytic investigation of anisotropy crossover, or corrections to scaling, and dilution induced criticality in diffusion dynamics for some exactly soluble lattice based systems.

Such critical, crossover and corrections-to-scaling effects in dynamics are conveniently described in phenomenological scaling terms [23]. Moreover, scaling methods, which are designed to exploit the diverging characteristic length which leads to criticality, can now be set out in such a way that they can handle dynamic situations directly in position space [5], which has unique advantages for treating random systems [5, 8, 24] and non-random, non-uniform systems such as fractals [8-11] (where they provide the only route to an exact solution).

In this letter such scaling methods are used to provide exact treatments of anisotropic diffusion in two such systems. The first is a linear chain subject to simple bias and weak dilution. The usual diffusion relationship $\omega = Dk^2$ is found to be modified by both the bias, which leads to the physically expected drift effects characterised by a modified exponent, and by dilution. The second system to be treated is a (pure) gasket fractal which can be subjected to more general forms of lattice anisotropy. It is shown that bias again causes crossover to drift behaviour, with crossover exponent unity, and rotational anisotropies cause corrections to scaling.

The discrete diffusion equation for a linear chain with nearest-neighbour hopping probabilities different in the two directions (i.e. bias) is

$$(1-\omega)u_n = x(1+x)^{-1}u_{n+1} + (1+x)^{-1}u_{n-1} \tag{1}$$

where ω is a reduced frequency, u_n characterises the occupation probability for site n, and the two x-dependent coefficients are the hopping probabilities, so $x-1$ is a measure of the bias. Dilution is ignored for the present. A decimation scaling [6] of the equation of motion (1) is simply achieved by writing down corresponding equations with $n \to n \pm 1$ and using them to eliminate $u_n \pm 1$. This results in an equation of similar form to (1) but now relating u_n to $u_n \pm 2$. The resulting equation has in place of ω, x new variables ω', x' given by

$$\omega' = (1+x)^2(1+x^2)^{-1}(2\omega - \omega^2) \tag{2}$$

$$x' = x^2. \tag{3}$$

These are the transformation equations corresponding to a length scaling of the pure system by a dilatation factor $b = 2$. Now it is necessary to generalise the above decimation procedure to the dilute case. Here the hopping terms ('bonds') are random variables subject to an initial binary distribution corresponding to the following two possibilities: bond present or absent with probability p or $(1-p)$ where p is the bond concentration. The decimation process then leads to new random bond variables, corresponding to the effective hoppings between sites n and $n \pm 2$, and hence to a scaling of their distribution exactly analogous to that found in a treatment of dynamics of the diluted Heisenberg chain [24]. Though in general correlations develop, near the pure limit they are negligible and the result is that (2) and (3) remain correct except for terms of order $(1-p)$, while in addition p scales according to the exact equation

$$p' = p^2. \tag{4}$$

To obtain the crossover induced in the dynamics in the diffusive limit ($\omega \to 0$) by weak dilution and anisotropy it is sufficient to linearise the scaling equations about the pure, isotropic, zero-frequency fixed point $(\omega^*, x^*, p^*) \equiv (0, 1, 1)$. This results in a dynamic scaling form for the characteristic diffusion frequency, together with an associated critical behaviour of the percolation correlation length ξ:

$$\omega = k^z G((\delta x)^{\phi}/k, k\xi), \qquad \xi \propto (\delta p)^{-\nu}. \tag{5}$$

In these expressions, $\delta x \equiv x-1$, $\delta p \equiv 1-p$, and k, are all small, and the dynamic exponent, anisotropy crossover exponent, and percolation exponent take the values $z=2$, $\phi=1$, $\nu=1$. The above results are exact.

The asymptotic forms of the scaling function $G(\alpha, \beta)$ are $G \sim D$ for $\beta \gg 1 \gg \alpha$; $G \sim A\alpha$ for $\alpha, \beta \gg 1$; $G \sim B\beta^{-2}$ for $1 \gg \alpha, \beta$; $G \sim C\alpha\beta^{-1}$ for $\alpha \gg 1 \gg \beta$. D is the diffusion constant, and A, B, C are three other constants (with $B \sim D$, $A \sim C$). The first three of these forms imply that the bias causes the low-frequency dispersion to cross over from quadratic ($\omega \sim Dk^2$) to linear form $\omega \sim Ak\delta x$, and the dilution causes crossover to $\omega \sim B/\xi^2$, which is characteristic [5-8, 24] of localised diffusion on clusters of characteristic size ξ. When both dilution and bias are present the asymptotic behaviour is $\omega \sim C\delta x/\xi$, which can be interpreted as due to ballistic motion limited by the finite cluster size.

The second system to be discussed is the triangular Sierpinski gasket fractal [12]. This has been proposed as a model for the infinite cluster backbone at the percolation threshold [2, 13] and that is one motivation for considering it here. Its static and isotropic dynamic scaling properties have been the subject of much recent work [8-11, 13]. A generalisation to anisotropic diffusion dynamics is now presented, which provides exact statements for anisotropy effects in the model.

The fractal is obtained from an equilateral triangle by dividing it into four equal triangles, discarding the central one, and continuing the same process of division, etc, for each remaining triangle [12, 13]. A part of the resulting fractal is shown in figure 1(a). As indicated in figure 1(b), α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$ are used to denote reduced hopping probabilities (each incorporating a frequency-dependent factor) different in the six hexagonal directions. With this notation the diffusion equation analogous to (1), for hopping to and from the vertex 0 of figure 1(a), is

$$u_0 = \alpha u_1 + \bar{\gamma} u_2 + \beta u_3 + \bar{\alpha} u_4. \tag{6}$$

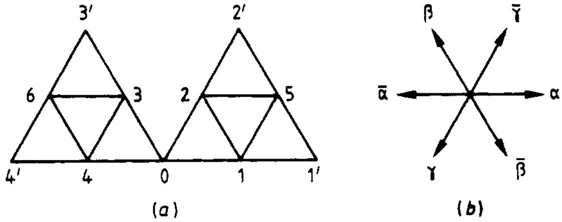

Figure 1. (a) Part of fractal used for decimation process; (b) anisotropic hopping variables.

The scaling of frequency ω and anisotropy is obtained by constructing the transformations of the variables $\alpha, \ldots, \bar{\gamma}$ by a decimation process similar to that used for the chain, now eliminating u_1, u_2, u_3, u_4, u_5, u_6 using their equations of motion. This achieves a length scaling of the fractal by $b=2$. The process involves the inversion of 3×3 matrices, because the variables u_1, u_2, u_5 satisfy coupled equations, as do u_3, u_4, u_6. The coefficients of $u_{1'}$, $u_{2'}, \ldots$ in the resulting scaled form of (6) then give the transformed variables α', $\bar{\gamma}'$, β', $\bar{\alpha}'$ as functions of the original variables. The (D_3) symmetries of the lattice, under rotations by $2\pi/3$ or reflections, enable the transformed variables $\bar{\beta}'$, γ' to be obtained and also require, for consistency with the lattice symmetries, that the variables must (before and after scaling) be related by

$$\alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} (\equiv \lambda). \tag{7}$$

Thus only four independent variables occur. A convenient choice is α, β, γ, λ, where λ is defined by (7).

The resulting exact transformations are

$$\alpha' = \alpha^2\Delta/\Gamma, \qquad \beta' = \beta^2\Delta/\Gamma, \qquad \gamma' = \gamma^2\Delta/\Gamma, \qquad \lambda' = \lambda^2\Delta\bar{\Delta}/\Gamma^2 \tag{8}$$

where, with $\mu \equiv \alpha\beta\gamma$, $\bar{\mu} \equiv \lambda^3/\mu$,

$$\begin{aligned}\Gamma &= 1 - 7\lambda - 3\mu - 3\bar{\mu},\\ \Delta &= 1 + \lambda + 3\lambda^2/\mu + \lambda^4/\mu^2 \equiv \Delta(\lambda, \mu), \qquad \bar{\Delta} = \Delta(\lambda, \bar{\mu}).\end{aligned} \tag{9}$$

The anisotropy-induced crossover in the diffusive dynamics is obtained from the transformation (8) linearised about its isotropic zero-frequency fixed point

$$\alpha^* = \beta^* = \gamma^* = \lambda^{*1/2} = \tfrac{1}{4}. \tag{10}$$

The eigenvalues (λ_ω, λ_d, λ_r) and associated eigenvectors ($\alpha - \alpha^*$, $\bar{\alpha} - \bar{\alpha}^*, \ldots, \bar{\gamma} - \bar{\gamma}^*$) are

$$\lambda_\omega = 5: \quad (1, 1, 1, 1, 1, 1) \tag{11a}$$

$$\lambda_d = 2: \quad (a, -a, b, -b, c, -c), \quad a + b + c = 0 \tag{11b}$$

$$\lambda_r = \tfrac{3}{5}; \quad (1, -1, 1, -1, 1, -1). \tag{11c}$$

These eigenvectors correspond respectively to the isotropic case (11*a*), the biased case (11*b*) in which the hopping is uniaxially anisotropic, and a rotationally anisotropic case (11*c*) tending to give rotational diffusion. The associated exponents are

$$z = \log_2 5, \qquad \phi = \log_2 2 = 1, \qquad \zeta = \log_2 \tfrac{5}{3}. \tag{12}$$

z is the 'anomalous' dynamic critical exponent for the isotropic case [8-11] and ϕ and ζ are new exponents respectively characterising the crossover to the drift behaviour caused by bias, and the corrections-to-scaling caused by the effects of rotational anisotropy, which is an irrelevant operator.

The dynamic scaling form analogous to (5), but (because of the lack of translational invariance in the fractal) now more appropriately written as a relation between diffusion length R and diffusion time t is, for large R, t,

$$R = t^{1/z}F(dt^{\phi/z}, rt^{-\zeta/z}) \qquad (d, r \to 0) \tag{13}$$

where d and r are measures of bias and rotational anisotropy respectively. For small argument F goes to a constant, giving isotropic diffusion behaviour. Other asymptotic behaviours can be obtained by considering the neighbourhood of the anisotropic fixed points of (8). Of these, the most important ones are those governing the new asymptotic drift behaviour, and they have typically one hopping variable non-zero. The associated dynamic eigenvalue is 2, so the bias-induced crossover is to asymptotic behaviour $R \propto dt$ (of strikingly simple form for a fractal result).

The treatment just given involves a generalised diffusion equation, (6), the variables of which have only been restricted by symmetry. In a normal lattice no further restrictions arise from making the total hopping probability to and from all sites the same, but in the anisotropic fractal such a further requirement limits the possible anisotropy only to rotational type and suppresses the drift crossover. Since this is due to a special characteristic of the fractal this unnecessarily restricted viewpoint has been avoided here.

The scaling treatment given here of diffusion in non-uniform (random or fractal) systems has indicated the possible effects of anisotropy on the diffusive limit behaviour. The relevance or irrelevance of anisotropy appears to depend on the nature of the anisotropy rather than whether it accompanies random or fractal aspects, whose predominant effects are to give additional (dilution-induced) crossover (equation (5)), or exotic exponents (equation (12)) which model those at the percolation threshold.

Whether the unit value of the crossover exponent ϕ in all the cases considered (including the fractal, where integer exponents are uncommon) is a coincidence or has significance needs further investigation. Similarly, though the crossover to drift behaviour seen in the biased cases is in qualitative agreement with Monte Carlo observations [21], for a quantitative comparison a direct scaling treatment of the random higher-dimensional system, as in reference [9] but with bias, is needed. There is so far no simulation with which the anisotropic fractal results can be compared. It would be of considerable interest to have both simulations and more complete theoretical discussions (e.g. on a real random system) of diffusion with such anisotropies.

References

[1] Enz C P (ed) 1979 *Dynamic Critical Phenomena and Related Topics* (Berlin: Springer)
[2] Kirkpatrick S 1979 in *Ill-Condensed Matter* ed R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland)
[3] Korenblit I Ya and Shender E F 1979 *Usp. Fiz. Nauk* **126** 233 (1978 *Sov. Phys.-Usp.* **21** 832)
[4] Stinchcombe R B 1983 in *Phase Transitions and Critical Phenomena* vol 7 ed C Domb and J L Lebowitz (London: Academic) p 151
[5] Stinchcombe R B 1985 in *Highlights of Condensed Matter Physics* (*Proc. Varenna Summer School*) ed F Bassani, F Fumi and M P Tosi (Amsterdam: North-Holland)
[6] Ziman T A L 1979 *J. Phys. C: Solid State Phys.* **12** 2645
[7] Kumar D 1982 *Phys. Rev.* B **25** 2069
[8] Harris C K and Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 1399
[9] Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
[10] Rammal R and Toulouse G 1983 *J. Physique Lett.* **44** L13
[11] Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
[12] Mandelbrot B B 1977 *Fractals* (San Francisco: Freeman).
[13] Gefen Y, Aharony A, Mandelbrot B and Kirkpatrick S 1980 *Phys. Rev. Lett.* **45** 855
[14] Pandey R B and Stauffer D 1983 *Phys. Rev. Lett.* **51** 527
[15] Lewis S J and Stinchcombe R B 1984 *Phys. Rev. Lett.* **52** 1021
[16] Fisher M E 1974 *Rev. Mod. Phys.* **46** 597
[17] Stanley H E, Kang K, Redner S and Blumberg R 1983 *Phys. Rev. Lett.* **51** 1223
[18] Redner S and Kang K 1983 *Phys. Rev. Lett.* **51** 1729
[19] Amit D J, Parisi G and Peliti L 1983 *Phys. Rev.* B **27** 1935
[20] Webman I 1984 *Phys. Rev. Lett.* **52** 220
[21] Pandey R B to be published
[22] Khantha M and Balakrishnan V 1985 *Phys. Rev.* B to be published
[23] Halperin B I and Hohenberg P C 1969 *Phys. Rev.* **117** 952
[24] Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 200